

Homological Algebra Seminar Week 2

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1 Abelian categories

Abelian categories generalize some very useful features of the category of abelian groups. In particular, morphisms in abelian categories can be added and possess the notions of kernel and cokernel with the desired properties. Thanks to these properties, the discipline of homological algebra becomes a powerful tool, with direct applications in areas such as topology or algebraic geometry.

Intuitively, abelian categories resemble the category of abelian groups $\mathbf{Ab}$. Apart from $\mathbf{Ab}$ itself, some examples of abelian categories include the category $R\text{-}\mathbf{Mod}$ of left (equivalently right) modules over a given ring R , the category $\mathbb{k}\text{-}\mathbf{Vect}$ of vector spaces over a field $\mathbb{k}$ or the category $\mathbf{Ab}_{\text{fin}}$ of finitely generated abelian groups.

We will need some preliminary definitions before getting to the concept of abelian category.

1.1 Definitions and basic properties

Definition 1.1. A category $\mathcal{A}$ is called a *preadditive category* or **Ab-category** if, for every pair of objects A and B in $\mathcal{A}$, the hom-set $\text{Hom}_{\mathcal{A}}(A, B)$ has the structure of an abelian group where composition distributes over addition. This means, given a diagram of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{g'} \end{array} C \xrightarrow{h} D$$

then $f(g + g')h = fgh + fg'h$.

Notice that, in an **Ab-category**, the hom-set $\text{Hom}_{\mathcal{A}}(A, A)$ has the structure of an associative ring.

Definition 1.2. Given two **Ab-categories** $\mathcal{A}, \mathcal{B}$, a functor $F : \mathcal{B} \rightarrow \mathcal{A}$ is called an *additive functor* if, for each pair of objects B, B' in $\mathcal{B}$, the map

$$\text{Hom}_{\mathcal{B}}(B, B') \longrightarrow \text{Hom}_{\mathcal{A}}(F(B), F(B'))$$

is a group homomorphism, i.e. $F(f + f') = F(f) + F(f')$ for $f, f' : B \rightarrow B'$.

Definition 1.3. An **Ab-category** $\mathcal{A}$ is called an *additive category* if it has an initial object 0 and a product $A \times B$ for each pair of objects A, B in $\mathcal{A}$.

Proposition 1.4. *In an additive category, the coproduct of any two objects exists and is isomorphic to their product.*

Proof. Let $\mathcal{A}$ be an additive category and A, B two objects in $\mathcal{A}$. Consider the product $A \times B$ and the maps

$$A \xrightarrow{\alpha} A \times B \xleftarrow{\beta} B$$

given by $\alpha = \text{id}_A \times 0$ and $\beta = 0 \times \text{id}_B$. Let us see that this defines a coproduct. Given an object C and maps $f : A \rightarrow C$ and $g : B \rightarrow C$, we seek for a unique $h : A \times B \rightarrow C$ such that $h\alpha = f$ and $h\beta = g$. Set $h = f \circ \pi_A + g \circ \pi_B$. Then, one checks that $h\alpha = (f\pi_A + g\pi_B)(\text{id}_A \times 0) = f\text{id}_A + g0 = f$, and equivalently $h\beta = g$. Uniqueness follows by observing that if $f = \text{id}_A$, one needs $h\alpha = \text{id}_A$, and similarly with $g = \text{id}_B$. Thus, $A \times B$ satisfies the universal property of coproducts. $\square$

Using induction, the statement follows for any finite coproduct.

Definition 1.5. An *abelian category* is an additive category $\mathcal{A}$ satisfying the following properties:

1. Every map in $\mathcal{A}$ has a kernel and a cokernel.
2. Every monic in $\mathcal{A}$ is the kernel of its cokernel.
3. Every epi in $\mathcal{A}$ is the cokernel of its kernel.

Remark 1.6. In the abelian categories mentioned at the start of the section, the kernel and cokernel of a given map are the usual ones in group, module or vector space homomorphisms, respectively. However, we notice that **Groups** is not an abelian category: it suffices to take the inclusion map $H \hookrightarrow G$ of a non-normal subgroup H of a group G . This is a monic map, but it can never be a kernel, as kernels are normal subgroups.

In particular, in an abelian category monics are kernels and epis are cokernels. Next, we can also define the notion of image of a morphism, by replicating the construction in abelian groups. In this way, the resulting object will match the image set in the case that the corresponding objects contain underlying sets.

Given a morphism $f : A \rightarrow B$, consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & \text{coker } f \\ & \searrow & \uparrow & & \\ & & \text{ker}(\text{coker } f) & & \end{array}$$

where the arrows $B \rightarrow \text{coker } f$ and $\text{ker}(\text{coker } f) \rightarrow B$ are given by definition of kernel and cokernel. By the universal property of the kernel, there exists a unique $A \rightarrow \text{ker}(\text{coker } f)$. In abelian groups, given that $\text{coker } f \cong B/\text{im } f$, this

is exactly the codomain restriction of f to its image $\text{im } f$. Hence, we define the object

$$\text{im } f := \ker(\text{coker } f).$$

The next result has been proved in Exercise 1(c).

Proposition 1.7. *The map $A \rightarrow \text{im } f$ is epi.*

Once we have defined the image, the notion of exactness arises.

Definition 1.8. A sequence of arrows $A \xrightarrow{f} B \xrightarrow{g} C$ is called *exact* (at B) if $g \circ f = 0$ and the canonical morphism $\text{im } f \rightarrow \ker g$ is an isomorphism.

Definition 1.9. A subcategory $\mathcal{B} \subseteq \mathcal{A}$ is called an *abelian subcategory* if

1. $\mathcal{B}$ is abelian.
2. Every exact sequence in $\mathcal{B}$ is also exact in $\mathcal{A}$.

Definition 1.10. An additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$ is said to be *left-exact* (resp. *right-exact*) if, for every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{A}$, the sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$$

(resp. the sequence $F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$) is exact in $\mathcal{B}$.

A functor is called *exact* if it is simultaneously left- and right-exact.

Definition 1.11. A contravariant additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *left-exact* (resp. *right-exact*) if $F^{\text{op}} : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ is left-exact (resp. right).

Let us now see a couple of non-trivial examples of abelian categories.

Example 1.12.

1. Let R be a commutative ring, and consider the subcategory $R\text{-}\mathbf{Mod}_{\text{fin}}$ of finitely generated R -modules. This category is abelian if and only if R is a Noetherian ring. Indeed, if R is Noetherian, the kernel and image of a morphism are finitely generated submodules of the corresponding modules, and thus lie in the category. This need not hold if R is not Noetherian, e.g. $R = K[X_1, X_2, \dots]$ for some field K . For instance, the kernel of the morphism $K[X_1, X_2, \dots] \rightarrow K$ mapping each $X_i \mapsto 0$ is the submodule $(X_1, X_2, \dots)$, which is not finitely generated.
2. Given a category I and an abelian category $\mathcal{A}$, the functor category $\mathcal{A}^I$ is also an abelian category.

The next result presents characterizations of additive and abelian categories. Its proof is left to the reader

Proposition 1.13. *Let $\mathcal{C}$ be a full subcategory of an abelian category $\mathcal{A}$. Then:*

1. *$\mathcal{C}$ is additive if and only if 0 is in $\mathcal{C}$ and $\mathcal{C}$ is closed under products.*
2. *$\mathcal{C}$ is abelian and $\mathcal{C} \hookrightarrow \mathcal{A}$ is exact if and only if $\mathcal{C}$ is additive and closed under kernels and cokernels.*

Finally, we consider the following important example of a left-exact functor.

Proposition 1.14. *Let $\mathcal{A}$ be an abelian category and M an object in $\mathcal{A}$. Then $\text{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \rightarrow \mathbf{Ab}$ is a left-exact functor.*

Proof. Consider an exact sequence $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ in $\mathcal{A}$. Let us check that

$$0 \rightarrow \text{Hom}(M, A) \xrightarrow{\alpha_*} \text{Hom}(M, B) \xrightarrow{\beta_*} \text{Hom}(M, C)$$

is also exact. First, at $\text{Hom}(M, A)$, consider $\gamma \in \text{Hom}(M, A)$ such that $\alpha_*\gamma = \alpha \circ \gamma = 0$. Then, as α is monic by exactness, $\gamma = 0$. Thus, α_* is also monic. Then, at $\text{Hom}(M, B)$, it is clear that $(\beta_* \circ \alpha_*)(\gamma) = \beta \circ \alpha \circ \gamma$ for any $\gamma \in \text{Hom}(M, A)$. It remains to check that $\ker \beta_* \subseteq \text{im } \alpha_*$ as subgroups of $\text{Hom}(M, B)$. For that, take $\delta \in \text{Hom}(M, B)$ such that $\beta_*\delta = \beta \circ \delta = 0$. Then, since $A \cong \ker(\beta)$ by exactness of the original sequence, the universal property of kernels implies that there exists a (unique) $\gamma \in \text{Hom}(M, A)$ such that $\alpha \circ \gamma = \delta$, as in the diagram.

$$\begin{array}{ccccc} & & A \cong \ker \beta & & \\ & \nearrow \gamma & \downarrow \alpha & & \\ M & \xrightarrow{\delta} & B & \xrightarrow{\beta} & C \end{array}$$

Thus, $\delta \in \text{im } \alpha_*$ as we wished. $\square$

Corollary 1.15. *The functor $\text{Hom}_{\mathcal{A}}(-, M)$ is a left-exact contravariant functor.*

Proof. It suffices to notice that $\text{Hom}_{\mathcal{A}}(A, M) = \text{Hom}_{\mathcal{A}^{\text{op}}}(M, A)$ for any A in $\mathcal{A}$, as arrows are reversed in $\mathcal{A}^{\text{op}}$. The statement then follows from the previous proposition. $\square$

1.2 Freyd-Mitchell embedding theorem

The following result provides an inclusion of any abelian category into a category of modules over a certain ring, preserving the underlying categorical structure. This is a very powerful result, as it allows to regard objects in abelian categories as modules, thus possessing the numerous well-known algebraic properties of these.

For its statement, recall that a category is called *small* if the class of all its objects is a set. Furthermore, in the next pages we will work with the *Yoneda embedding* $h : \mathcal{A} \rightarrow \mathbf{Ab}^{\mathcal{A}}$, defined on any abelian category $\mathcal{A}$. This is given by $h(A) = \text{Hom}_{\mathcal{A}}(-, A)$ for each object A in $\mathcal{A}$.

Theorem 1.16. *Let $\mathcal{A}$ be a small abelian category. Then, there exists a ring R and an exactly fully faithful functor $i : \mathcal{A} \hookrightarrow R\text{-Mod}$.*

Proof (sketch). The Yoneda embedding $h : \mathcal{A} \rightarrow \mathbf{Ab}^{\mathcal{A}}$ is a left-exact functor, but not right-exact in general. It factors through the category $\mathcal{L}$ of left-exact functors $\mathcal{A}^{\text{op}} \rightarrow \mathbf{Ab}$. This is an abelian category, and the functor $\mathcal{A} \rightarrow \mathcal{L}$ is actually exact.

It can be shown that $\mathcal{L}$ is equivalent to $R\text{-Mod}$ for $R := \text{Hom}_{\mathcal{L}}(P, P)$, where P is some special object. $\square$

Example 1.17. Consider $\mathcal{A}$ to be the category of $\mathbb{Z}$ -graded R -modules for a ring R . Then $\mathcal{A}$ injects into the category of $\prod_{i \in \mathbb{Z}} R$ -modules of the form $\bigoplus_{i \in \mathbb{Z}} M_i$.

1.3 The Yoneda lemma

The *Yoneda lemma* is one fundamental result in category theory, providing a representation of any small category $\mathcal{C}$ via functors from $\mathcal{C}$ to $\mathbf{Set}$. Here, we state a special instance of the main result.

Theorem 1.18. *The Yoneda embedding $h : \mathcal{A} \rightarrow \mathbf{Ab}^{\mathcal{A}}$ reflects exactness, i.e. a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{A}$ if for all M in $\mathcal{A}$, the sequence*

$$\text{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\alpha_*} \text{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\beta_*} \text{Hom}_{\mathcal{A}}(M, C)$$

is exact.

Proof. Consider a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathcal{A}$. First, we may take $M = A$ in the hypothesis. By exactness of the sequence in $\mathbf{Ab}^{\mathcal{A}}$, we have

$$0 = (\beta_* \circ \alpha_*)(\text{id}_A) = \beta \circ \alpha \circ \text{id}_A = \beta \circ \alpha.$$

It remains to show that the universal map between $\text{im } \alpha$ and $\ker \beta$ is an isomorphism. In this case, pick $M = \ker \beta$ with $i : \ker \beta \rightarrow B$. Since $\beta \circ i = \beta_* i = 0$, then $i \in \text{im } \alpha_*$, i.e. there exists $\gamma : \ker \beta \rightarrow A$ such that $i = \alpha_* \gamma = \alpha \circ \gamma$. Next, let us consider the construction of $\text{im } \alpha$ as in the definition, giving rise to arrows $A \xrightarrow{\alpha'} \text{im } \alpha \xrightarrow{j} B$. Given that $\beta \circ j \circ \alpha' = \beta \circ \alpha = 0$ and α' is epi, then $\beta \circ j = 0$ as well. Hence, by the universal property of the kernel, there exists $\delta : \text{im } \alpha \rightarrow \ker \beta$ such that $j = i \circ \delta$. This is displayed in the following diagram.

$$\begin{array}{ccccc}
 & & \ker \beta & & \\
 & \swarrow \gamma & \downarrow i & \nwarrow & \\
 A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\
 & \searrow \alpha' & \uparrow j & \swarrow \delta & \\
 & & \text{im } \alpha & &
 \end{array}$$

Finally, notice that $j\alpha'\gamma\delta = \alpha\gamma\delta = i\delta = j$. Since j is monic, this implies $\alpha'\gamma\delta = \text{id}_{\text{im } \alpha}$. A parallel argument leads to $\delta\alpha'\gamma = \text{id}_{\ker \beta}$, and we deduce that δ is an isomorphism, as we wanted to show. $\square$

Remark 1.19. The converse of the Yoneda lemma is not true in general. For example, consider the exact sequence $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ in the category of abelian groups. Taking $M = \mathbb{Z}/2\mathbb{Z}$, one gets the following sequence between hom-sets

$$\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\pi_*} \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

However, π_* cannot be surjective because $\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ is the trivial group while $\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus note that, in Proposition ??, we need the injectivity of α for the resulting sequence to be exact at the middle position.

This allows to state the following result between adjoint functors, proved in Exercise 3. Recall the definition of an adjoint pair of functors: given categories $\mathcal{A}$ and $\mathcal{B}$, a pair of functors $L : \mathcal{A} \rightarrow \mathcal{B}$ and $R : \mathcal{B} \rightarrow \mathcal{A}$ is said to be adjoint if there is a natural isomorphism $\text{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(A, R(B))$.

Corollary 1.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $L : \mathcal{A} \rightarrow \mathcal{B}$ and $R : \mathcal{B} \rightarrow \mathcal{A}$ be an adjoint pair of functors. Then L is right-exact and R is left-exact.*